V. *On the Theory of Resonance.* *By the Hon.* J. W. Strutt, *M.A., Fellow of Trinity College, Cambridge.* *Communicated by* W. Spottiswoode, *F.R.S.*

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Introduction.

Although the theory of aërial vibrations has been treated by more than one generation of mathematicians and experimenters, comparatively little has been done towards obtaining a clear view of what goes on in any but the more simple cases. The extreme difficulty of any thing like a general deductive investigation of the question is no doubt one reason. On the other hand, experimenters on this, as on other subjects, have too often observed and measured blindly without taking sufficient care to simplify the conditions of their experiments, so as to attack as few difficulties as possible at a time. The result has been vast accumulations of isolated facts and measurements which lie as a sort of dead weight on the scientific stomach, and which must remain undigested until theory supplies a more powerful solvent than any now at our command. The motion of the air in cylindical organ-pipes was successfully investigated by Bernoulli and Euler, at least in its main features; but their treatment of the question of the open pipe was incomplete, or even erroneous, on account of the assumption that at the open end the air remains of invariable density during the vibration. Although attacked by many others, this difficulty was not finally overcome until Helmholtz †, in a paper which I shall have repeated occasion to refer to, gave a solution of the problem under certain restrictions, free from any arbitrary assumptions as to what takes place at the open end. Poisson and Stokes‡ have solved the problem of the vibrations communicated to an infinite mass of air from the surface of a sphere or circular cylinder. The solution for the sphere is very instructive, because the vibrations outside any imaginary sphere enclosing vibrating bodies of any kind may be supposed to take their rise in the surface of the sphere itself.

More important in its relation to the subject of the present paper is an investigation by Helmholtz of the air-vibrations in cavernous spaces (*Hohlraüme*), whose three dimensions are very small compared to the wave-length, and which communicate with the external atmosphere by small holes in their surfaces. If the opening be circular of area σ, and if S denote the volume, n the number of vibrations per second in the fundamental

* Additions made since the paper was first sent to the Royal Society are inclosed in square brackets [ ].

† Theorie der Luftschwingungen in Röhren mit offenen Enden. Crelle, 1860.

‡ Phil. Trans. 1868, or Phil. Mag. Dec. 1868.

note, and a the velocity of sound,

$$n=\frac{a\sigma^{\frac{1}{4}}}{2^{\frac{1}{2}}\pi^{\frac{5}{4}}S^{\frac{1}{2}}}.$$

HELMHOLTZ's theory is also applicable when there are more openings than one in the side of the vessel.

In the present paper I have attempted to give the theory of vibrations of this sort in a more general form. The extension to the case where the communication with the external air is no longer by a mere hole in the side, but by a neck of greater or less length, is important, not only because resonators with necks are frequently used in practice, but also by reason of the fact that the theory itself is applicable within wider limits. The mathematical reasoning is very different from that of HELMHOLTZ, at least in form, and will I hope be found easier. In order to assist those who may wish only for clear general ideas on the subject, I have broken up the investigation as much as possible into distinct problems, the results of which may in many cases be taken for granted without the rest becoming unintelligible. In Part I. my object has been to put what may be called the dynamical part of the subject in a clear light, deferring as much as possible special mathematical calculations. In the first place, I have considered the general theory of resonance for air-spaces confined nearly all round by rigid walls, and communicating with the external air by any number of passages which may be of the nature of necks or merely holes, under the limitation that both the length of the necks and the dimensions of the vessel are very small compared to the wave-length. To prevent misapprehension, I ought to say that the theory applies only to the fundamental note of the resonators, for the vibrations corresponding to the overtones are of an altogether different character. There are, however, cases of multiple resonance to which our theory is applicable. These occur when two or more vessels communicate with each other and with the external air by necks or otherwise; and are easily treated by LAGRANGE's general dynamical method, subject to a restriction as to the relative magnitudes of the wave-lengths and the dimensions of the system corresponding to that stated above for a single vessel. I am not aware whether this kind of resonance has been investigated before, either mathematically or experimentally. Lastly, I have sketched a solution of the problem of the open organ-pipe on the same general plan, which may be acceptable to those who are not acquainted with HELMHOLTZ's most valuable paper. The method here adopted, though it leads to results essentially the same as his, is I think more calculated to give an insight into the real nature of the question, and at the same time presents fewer mathematical difficulties. For a discussion of the solution, however, I must refer to HELMHOLTZ.

In Part II. the calculation of a certain quantity depending on the form of the necks of common resonators, and involved in the results of Part I., is entered upon. This quantity, denoted by c, is of the nature of a length, and is identical with what would be called in the theory of electricity the *electric conductivity* of the passage, supposed to be occupied by uniformly conducting matter. The question is accordingly similar to that of determining the electrical resistance of variously shaped conductors—an analogy of which I have not hesitated to avail myself freely both in investigation and statement. Much circumlocution is in this way avoided on account of the greater completeness of electrical phraseology. Passing over the case of mere holes, which has been already considered by HELMHOLTZ, and need not be dwelt upon here, we come to the value of the resistance for necks in the form of circular cylinders. For the sake of simplicity each end is supposed to be in an infinite plane. In this form the mathematical problem is definite, but has not been solved rigorously. Two limits, however (a higher and a lower), are investigated, between which it is proved that the true resistance must lie. The lower corresponds to a correction to the length of the tube equal to $\frac{\pi}{4}\times$(radius) for each end. It is a remarkable coincidence that HELMHOLTZ also finds the same quantity as an approximate correction to the length of an organ-pipe, although the two methods are entirely different and neither of them rigorous. His consists of an exact solution of the problem for an approximate cylinder, and mine of an approximate solution for a true cylinder; while both indicate on which side the truth must lie. The final result for a cylinder infinitely long is that the correction lies between ·785 R and ·828 R. When the cylinder is finite, the upper limit is rather smaller. In a somewhat similar manner I have investigated limits for the resistance of a tube of revolution, which is shown to lie between

$$\int\frac{dx}{\pi y^2}$$

and

$$\int\frac{dx}{\pi y^2}\left\{1+\frac{1}{2}\left(\frac{dy}{dx}\right)^2\right\},$$

where y denotes the radius of the tube at any point x along the axis. These formulæ apply whatever may be in other respects the form of the tube, but are especially valuable when it is so nearly cylindrical that $\frac{dy}{dx}$ is everywhere small. The two limits are then very near each other, and either of them gives very approximately the true value. The resistance of tubes, which are either not of revolution or are not nearly straight, is afterwards approximately determined. The only experimental results bearing on the subject of this paper, and available for comparison with theory, that I have met with are some arrived at by SONDHAUSS* and WERTHEIM†. Besides those quoted by HELMHOLTZ, I have only to mention a series of observations by SONDHAUSS‡ on the pitch of flasks with long necks which led him to the empirical formula

$$n=46705\frac{\sigma^{\frac{1}{2}}}{L^{\frac{1}{2}}S^{\frac{1}{2}}},$$

σ, L being the area and length of the neck, and S the volume of the flask. The corresponding equation derived from the theory of the present paper is

$$n=54470\frac{\sigma^{\frac{1}{4}}}{L^{\frac{1}{2}}S^{\frac{1}{2}}},$$

* POGG. Ann. vol. lxxxi. † Annales de Chimie, vol. xxxi. ‡ POGG. Ann. vol. lxxix.

which is only applicable, however, when the necks are so long that the corrections at the ends may be neglected—a condition not likely to be fulfilled. This consideration sufficiently explains the discordance. Being anxious to give the formulæ of Parts I. and II. a fair trial, I investigated experimentally the resonance of a considerable number of vessels which were of such a form that the theoretical pitch could be calculated with tolerable accuracy. The result of the comparison is detailed in Part III., and appears on the whole very satisfactory; but it is not necessary that I should describe it more minutely here. I will only mention, as perhaps a novelty, that the experimental determination of the pitch was not made by causing the resonators to speak by a stream of air blown over their mouths. The grounds of my dissatisfaction with this method are explained in the proper place.

[Since this paper was written there has appeared another memoir by Dr. SONDHAUSS* on the subject of resonance. An empirical formula is obtained bearing resemblance to the results of Parts I. and II., and agreeing fairly well with observation. No attempt is made to connect it with the fundamental principles of mechanics. In the Philosophical Magazine for September 1870, I have discussed the differences between Dr. SONDHAUSS's formula and my own from the experimental side, and shall not therefore go any further into the matter on the present occasion.]

PART I.

The class of resonators to which attention will chiefly be given in this paper are those where a mass of air confined almost all round by rigid walls communicates with the external atmosphere by one or more narrow passages. For the present it may be supposed that the boundary of the principal mass of air is part of an oval surface, nowhere contracted into any thing like a narrow neck, although some cases not coming under this description will be considered later. In its general character the fundamental vibration of such an air-space is sufficiently simple, consisting of a periodical rush of air through the narrow channel (if there is only one) into and out of the confined space, which acts the part of a reservoir. The channel spoken of may be either a mere hole of any shape in the side of the vessel, or may consist of a more or less elongated tube-like passage.

If the linear dimension of the reservoir be small as compared to the wave-length of the vibration considered, or, as perhaps it ought rather to be said, the quarter wave-length, the motion is remarkably amenable to deductive treatment. Vibration in general may be considered as a periodic transformation of energy from the potential to the kinetic, and from the kinetic to the potential forms. In our case the kinetic energy is that of the air in the neighbourhood of the opening as it rushes backwards or forwards. It may be easily seen that relatively to this the energy of the motion inside the reservoir is, under the restriction specified, very small. A formal proof would require the assistance of the general equations to the motion of an elastic fluid, whose use I wish to avoid in

* POGG. Ann. 1870.

this paper. Moreover the motion in the passage and its neighbourhood will not differ sensibly from that of an incompressible fluid, and its energy will depend only on the rate of total flow through the opening. A quarter of a period later this energy of motion will be completely converted into the potential energy of the compressed or rarefied air inside the reservoir. So soon as the mathematical expressions for the potential and kinetic energies are known, the determination of the period of vibration or resonant note of the air-space presents no difficulty.

The motion of an incompressible frictionless fluid which has been once at rest is subject to the same formal laws as those which regulate the flow of heat or electricity through uniform conductors, and depends on the properties of the potential function, to which so much attention has of late years been given. In consequence of this analogy many of the results obtained in this paper are of as much interest in the theory of electricity as in acoustics, while, on the other hand, known modes of expression in the former subject will save circumlocution in stating some of the results of the present problem.

Let h_0 be the density, and ϕ the velocity-potential of the fluid motion through an opening. The kinetic energy or *vis viva*

$$=\frac{1}{2}h_0\iiint\left[\left(\frac{d\phi}{dx}\right)^2+\left(\frac{d\phi}{dy}\right)^2+\left(\frac{d\phi}{dz}\right)^2\right]dxdydz,$$

the integration extending over the volume of the fluid considered

$$=\frac{1}{2}h_0\iint\phi\frac{d\phi}{dn}dS,$$

by GREEN'S theorem.

Over the rigid boundary of the opening or passage, $\frac{d\phi}{dn}=0$, so that if the portion of fluid considered be bounded by two equipotential surfaces, ϕ_1 and ϕ_2, one on each side of the opening,

$$vis\ viva=\frac{1}{2}h_0(\phi_1-\phi_2)\iint\frac{d\phi}{dn}dS=\frac{1}{2}h_0(\phi_1-\phi_2)\dot{X},$$

if $\dot{X}$ denote the rate of total flow through the opening.

At a sufficient distance on either side ϕ becomes constant, and the rate of total flow is proportional to the difference of its values on the two sides. We may therefore put

$$\phi_1-\phi_2=\frac{1}{c}\iint\frac{d\phi}{dn}dS=\frac{\dot{X}}{c},$$

where c is a linear quantity depending on the size and shape of the opening, and representing in the electrical interpretation the reciprocal of the *resistance* to the passage of electricity through the space in question, the specific resistance of the conducting matter being taken for unity. The same thing may be otherwise expressed by saying that c is the side of a cube, whose resistance between opposite faces is the same as that of the opening.

The expression for the *vis viva* in terms of the rate of total flow is accordingly

$$vis\ viva=\frac{h_0}{2}\frac{\dot{X}^2}{c}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If S be the capacity of the reservoir, the condensation at any time inside it is given by

$\frac{X}{S}$, of which the mechanical value is

$$\frac{1}{2}h_0a^2\frac{X^2}{S}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

a denoting, as throughout the paper, the velocity of sound.

The whole energy at any time, both actual and potential, is therefore

$$\frac{h_0}{2}\frac{\dot{X}^2}{c}+\frac{h_0}{2}a^2\frac{X^2}{S}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and is constant. Differentiating with respect to time, we arrive at

$$\ddot{X}+\frac{a^2c}{S}X=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

as the equation to the motion, which indicates simple oscillations performed in a time

$$2\pi\div\sqrt{\frac{a^2c}{S}}.$$

Hence if n denote the number of vibrations per second in the resonant note,

$$n=\frac{a}{2\pi}\sqrt{\frac{c}{S}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The wave-length λ, which is the quantity most immediately connected with the dimensions of the resonant space, is given by

$$\lambda=\frac{a}{n}=2\pi\sqrt{\frac{S}{c}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

A law of SAVART, not nearly so well known as it ought to be, is in agreement with equations (5) and (6). It is an immediate consequence of the principle of dynamical similarity, of extreme generality, to the effect that *similar* vibrating bodies, whether they be gaseous, such as the air in organ-pipes or in the resonators here considered, or solid, such as tuning-forks, vibrate in a time which is directly as their linear dimensions. Of course the material must be the same in two cases that are to be compared, and the geometrical similarity must be complete, extending to the shape of the opening as well as to the other parts of the resonant vessel. Although the wave-length λ is a function of the size and shape of the resonator only, n or the position of the note in the musical scale depends on the nature of the gas with which the resonator is filled. And it is important to notice that it is on the nature of the gas in and near the opening that the note depends, and *not* on the gas in the interior of the reservoir, whose inertia does not come into play during vibrations corresponding to the fundamental note. In fact we may say that the mass to be moved is the air in the neighbourhood of the opening, and that the air in the interior acts merely as a spring in virtue of its resistance to compression. Of course this is only true under the limitation specified, that the diameter of the reservoir is small compared to the quarter wave-length. Whether this condition is fulfilled in the case of any particular resonator is easily seen, *à posteriori*, by calculating the value of λ from (6), or by determining it experimentally.

Several Openings.

When there are two or more passages connecting the interior of the resonator with the external air, we may proceed in much the same way, except that the equation of energy by itself is no longer sufficient. For simplicity of expression the case of two passages will be convenient, but the same method is applicable to any number. Let X_1, X_2 be the total flow through the two necks, c_1, c_2 constants depending on the form of the necks corresponding to the constant c in formula (6); then T, the *vis viva*, is given by

$$T=\frac{h_0}{2}\left(\frac{\dot{X}_1^2}{c_1}+\frac{\dot{X}_2^2}{c_2}\right),$$

the necks being supposed to be sufficiently far removed from one another not to *interfere* (in a sense that will be obvious). Further,

$$V=\text{Potential Energy}=\frac{1}{2}h_0a^2\frac{(X_1+X_2)^2}{S}.$$

Applying LAGRANGE's general dynamical equation, $\frac{d}{dt}\left(\frac{dT}{d\dot{\psi}}\right)-\frac{dT}{d\psi}=-\frac{dV}{d\psi}$,

we obtain

$$\left.\begin{aligned}\frac{\ddot{X}_1}{c_1}+\frac{a^2}{S}(X_1+X_2)=0,\\ \frac{\ddot{X}_2}{c_2}+\frac{a^2}{S}(X_1+X_2)=0\end{aligned}\right\} \quad \ldots \ldots \ldots \quad (7)$$

as the equations to the motion.

By subtraction,

$$\frac{\ddot{X}_1}{c_1}-\frac{\ddot{X}_2}{c_2}=0,$$

or, on integration,

$$\frac{X_1}{c_1}=\frac{X_2}{c_2} \quad \ldots \ldots \ldots \ldots \quad (8)$$

Equation (8) shows that the motions of the air in the two necks have the same period and are at any moment in the same phase of vibration. Indeed there is no essential distinction between the case of one neck and that of several, as the passage from one to the other may be made continuously without the failure of the investigation.

When, however, the separate passages are sufficiently far apart, the constant c for the system, considered as a single communication between the interior of the resonator and the external air, is the simple sum of the values belonging to them when taken separately, which would not otherwise be the case. This is a point to which we shall return later, but in the mean time, by addition of equations (7), we find

$$\ddot{X}_1+\ddot{X}_2+\frac{a^2}{S}(c_1+c_2)(X_1+X_2)=0,$$

so that

$$n=\frac{a}{2\pi}\sqrt{\frac{c_1+c_2}{S}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

If there be any number of necks for which the values of c are c_1, c_2, c_3, ..., and no two of which are near enough to interfere, the same method is applicable, and gives

$$n=\frac{a}{2\pi}\sqrt{\frac{c_1+c_2+c_3+\ldots\ldots}{S}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9')$$

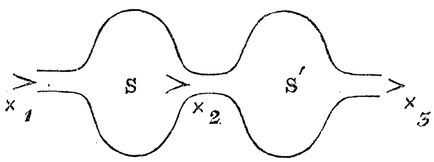
when there are two similar necks $c_2=c_1$, and

$$n=\sqrt{2}\times\frac{a}{2\pi}\sqrt{\frac{c}{S}}.$$

The note is accordingly higher than if there were only one neck in the ratio of $\sqrt{2}:1$, a fact observed by SONDHAUSS and proved theoretically by HELMHOLTZ for the case of openings which are mere holes in the sides of the reservoir.

Double Resonance.

Fig. 1.



Suppose that there are two reservoirs, S, S′, communicating with each other and with the external air by narrow passages or necks. If we were to consider S S′ as a single reservoir and to apply equation (9), we should be led to an erroneous result; for the reasoning on which (9) is founded proceeds on the assumption that, within the reservoir, the inertia of the air may be left out of account, whereas it is evident that the *vis viva* of the motion through the connecting passage may be as great as through the two others. However, an investigation on the same general plan as before meets the case perfectly. Denoting by X_1, X_2, X_3 the total flows through the three necks, we have for the *vis viva* the expression

$$T=\frac{1}{2}h_0\left\{\frac{\dot{X}_1^2}{c_1}+\frac{\dot{X}_2^2}{c_2}+\frac{\dot{X}_3^2}{c_3}\right\},$$

and for the potential energy

$$V=\frac{1}{2}h_0a^2\left\{\frac{(X_2-X_1)^2}{S}+\frac{(X_3-X_2)^2}{S'}\right\}.$$

An application of LAGRANGE's method gives as the differential equations to the motion,

$$\left.\begin{aligned}&\frac{\ddot{X}_1}{c_1}+a^2\frac{X_1-X_2}{S}=0,\\&\frac{\ddot{X}_2}{c_2}+a^2\left\{\frac{X_2-X_1}{S}+\frac{X_2-X_3}{S'}\right\}=0,\\&\frac{\ddot{X}_3}{c_3}+a^2\frac{X_3-X_2}{S'}=0.\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (10)$$

By addition and integration

$$\frac{X_1}{c_1}+\frac{X_2}{c_2}+\frac{X_3}{c_3}=0.$$

Hence, on elimination of X_2,

$$\left.\begin{aligned}&\ddot{X}_1+\frac{a^2}{S}\left\{(c_1+c_2)X_1+\frac{c_1c_2}{c_3}X_3\right\}=0,\\&\ddot{X}_3+\frac{a^2}{S'}\left\{(c_3+c_2)X_3+\frac{c_3c_2}{c_1}X_1\right\}=0.\end{aligned}\right\}$$

Assuming $X_1=A\varepsilon^{pt}$, $X_3=B\varepsilon^{pt}$, we obtain, on substitution and elimination of $A:B$,

$$p^4+p^2a^2\left\{\frac{c_1+c_2}{S}+\frac{c_3+c_2}{S'}\right\}+\frac{a^4}{SS'}\left\{c_1c_3+c_2(c_1+c_3)\right\}=0 \quad \ldots\ldots\ldots \quad (11)$$

as the equation to determine the resonant notes. If n be the number of vibrations per second, $n^2=-\frac{p^2}{4\pi^2}$, the values of p^2 given by (11) being of course both real and negative. The formula simplifies considerably if $c_3=c_1$, $S'=S$; but it will be more instructive to work this case from the beginning. Let $c_1=c_3=mc_2=mc$.

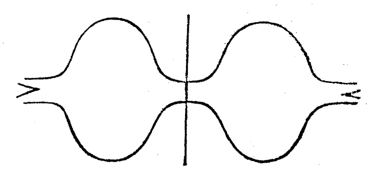
The differential equations take the form

$$\left.\begin{aligned}&\ddot{X}_1+\frac{a^2c}{S}\left\{(1+m)X_1+X_3\right\}=0,\\&\ddot{X}_3+\frac{a^2c}{S}\left\{(1+m)X_3+X_1\right\}=0,\end{aligned}\right\}\text{while } X_2=-\frac{X_1+X_3}{m}.$$

Hence

$$\left.\begin{aligned}&(X_1+X_3)^{\cdot\cdot}+\frac{a^2c}{S}(m+2)(X_1+X_3)=0,\\&(X_1-X_3)^{\cdot\cdot}+\frac{a^2c}{S}m(X_1-X_3)=0.\end{aligned}\right\}$$

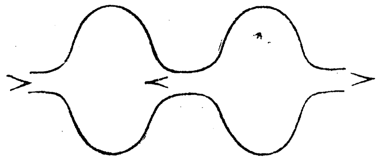
Fig. 2.



The whole motion may be regarded as made up of two parts, for the first of which $X_1+X_3=0$; which requires $X_2=0$. This motion is therefore the same as might take place were the communication between S and S′ cut off, and has its period given by

$$n^2=\frac{a^2c_1}{4\pi^2S}=\frac{a^2mc}{4\pi^2S}.$$

Fig. 3.



For the other component part, $X_1-X_3=0$, so that

$$X_2=-\frac{2X_1}{m}, \qquad n'^2=\frac{a^2(m+2)c}{4\pi^2 S} \quad . \quad . \quad . \quad (12)$$

Thus $\frac{n'^2}{n^2}=\frac{m+2}{m}$, which shows that the second note is the higher. It consists of vibrations in the two reservoirs opposed in phase and modified by the connecting passage, which acts in part as a second opening to both, and so raises the pitch. If the passage is small, so also is the difference of pitch between the two notes. A particular case worth notice is obtained by putting in the general equation $c_3=0$, which amounts to suppressing one of the communications with the external air. We thus obtain

$$p^4+a^2p^2\left(\frac{c_1+c_2}{S}+\frac{c_2}{S'}\right)+\frac{a^4}{SS'}c_1c_2=0\,;$$

or if $S=S'$, $c_1=mc_2=mc$,

$$p^4+a^2p^2\frac{c}{S}(m+2)+\frac{a^4c^2}{S^2}m=0,$$

$$n^2=\frac{a^2c}{8\pi^2S}\{m+2\pm\sqrt{m^2+4}\}.$$

If we further suppose $m=1$ or $c_2=c_1$,

$$n^2=\frac{a^2c}{8\pi^2S}(3\pm\sqrt{5}).$$

If N be the number of vibrations for a simple resonator (S, c),

$$N^2=\frac{a^2c}{4\pi^2S}\,;$$

$$\therefore\ n_1^2\div N^2=\frac{3+\sqrt{5}}{2}=2{\cdot}618,$$

$$N^2\div n_2^2=\frac{2}{3-\sqrt{5}}=2{\cdot}618.$$

It appears therefore that the interval from n_1 to N is the same as from N to n_2, namely, $\sqrt{2{\cdot}618}=1{\cdot}618$, or rather more than a fifth. It will be found that whatever the value of m may be, the interval between the resonant notes cannot be less than 2·414, which is about an octave and a minor third. The corresponding value of m is 2.

A similar method is applicable to any combination of reservoirs and connecting passages, no matter how complicated, under the single restriction as to the comparative magnitudes of the reservoirs and wave-lengths; but the example just given is sufficient to illustrate the theory of multiple resonance. In Part III. a resonator of this sort will be described, which was constructed for the sake of a comparison between the theory and experiment. In applying the formulæ (6) or (12) to an actual measurement, the question will arise whether the volume of the necks, especially when they are rather large, is to be included or not in S. At the moment of rest the air in the neck is compressed or rarefied as well as that inside the reservoir, though not to the same degree; in fact the condensation must vary continuously between the interior of the resonator and the external air. This consideration shows that, at least in the case of necks which are tolerably symmetrical, about half the volume of the neck should be included in S.

[In consequence of a suggestion made by Mr. CLERK MAXWELL, who reported on this paper, I have been led to examine what kind of effect would be produced by a deficient rigidity in the envelope which contains the alternately compressed and rarefied air. Taking for simplicity the case of a sphere, let us suppose that the radius, instead of remaining constant at its normal value R, assumes the variable magnitude $R+\varrho$. We have

$$\text{kinetic energy}=\frac{h_0\dot{X}^2}{2c}+\frac{m}{2}\dot{\varrho}^2,$$

$$\text{potential energy}=\frac{h_0a^2}{2S}\{X+4\pi R^2\varrho\}^2+\tfrac{1}{2}\beta\varrho^2,$$

where m and β are constants expressing the inertia and rigidity of the spherical shell. Hence, by LAGRANGE's method,

$$\left.\begin{aligned}&\ddot{X}+\frac{ca^2}{S}(X+4\pi R^2\varrho)=0,\\&m\ddot{\varrho}+4\pi R^2\frac{h_0a^2}{S}(X+4\pi R^2\varrho)+\beta\varrho=0,\end{aligned}\right\}$$

equations determining the periods of the two vibrations of which the system is capable. It might be imagined at first sight that a yielding of the sides of the vessel would necessarily lower the pitch of the resonant note; but this depends on a tacit assumption that the capacity of the vessel is largest when the air inside is most compressed. But it may just as well happen that the opposite is true. Everything depends on the relative magnitudes of the periods of the two vibrations supposed for the moment independent of one another. If the note of the shell be very high compared to that of the air, the inertia of the shell may be neglected, and this part of the question treated statically. Putting in the equations $m=0$, we see that the phases of X and ϱ are opposed, and then X goes through its changes more slowly than before. On the other hand, if it be the note of the air-vibration, which is much the higher, we must put $\beta=0$, which leads to

$$4\pi R^2h_0\ddot{X}-cm\ddot{\varrho}=0,$$

showing that the phases of X and ϱ agree. Here the period of X is diminished by the yielding of the sides of the vessel, which indeed acts just in the same way as a second aperture would do. A determination of the actual note in any case of a spherical shell of given dimensions and material would probably be best obtained deductively.

But in order to see what probability there might be that the results of Part III. on glass flasks were sensibly modified by a want of rigidity, I thought it best to make a direct experiment. To the neck of a flask was fitted a glass tube of rather small bore, and the whole filled with water so as to make a kind of water-thermometer. On removing by means of an air-pump the pressure of the atmosphere on the outside of the bulb, the liquid fell in the tube, but only to an extent which indicated an increase in the capacity of the flask of about a ten-thousandth part. This corresponds in the ordinary arrangement to a doubled density of the contained air. It is clear that so small a yielding could produce no sensible effect on the pitch of the air-vibration.]

Open Organ-pipes.

Although the problem of open organ-pipes, whose diameter is very small compared to their length and to the wave-length, has been fully considered by HELMHOLTZ, it may not be superfluous to show how the question may be attacked from the point of view of the present paper, more especially as some important results may be obtained by a comparatively simple analysis. The principal difficulty consists in finding the connexion between the spherical waves which diverge from the open end of the tube into free space, and the waves in the tube itself, which at a distance from the mouth, amounting to several diameters, are approximately plane. The transition occupies a space which is large compared to the diameter, and in order that the present treatment may be applicable must be small compared to the wave-length. This condition being fulfilled, the compressibility of the air in the space mentioned may be left out of account and the difficulty is turned. Imagine a piston (of infinitely small thickness) in the tube at the place where the waves cease to be plane. The motion of the air on the free side is entirely determined by the motion of the piston, and the *vis viva* within the space considered may be expressed by

$$\frac{1}{2}h_0\frac{\dot{X}^2}{c},$$

where X denotes the rate of total flow at the place of the piston, and c is, as before, a linear quantity depending on the form of the mouth. If Q is the section of the tube and ψ the velocity potential,

$$\dot{X}=Q\frac{d\psi}{dx}.$$

The most general expression for the velocity-potential of plane waves is

$$\psi=\left(\frac{A}{k}\sin kx+B\cos kx\right)\cos 2\pi nt+\beta\cos kx\sin 2\pi nt, \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$\frac{d\psi}{dx}=(A\cos kx-Bk\sin kx)\cos 2\pi nt-\beta k\sin kx\sin 2\pi nt,$$

where

$$k=\frac{2\pi}{\lambda}=\frac{2\pi n}{a}.$$

When $x=0$,

$$\left.\begin{aligned}\psi&=B\cos 2\pi nt+\beta\sin 2\pi nt,\\ \frac{d\psi}{dx}&=A\cos 2\pi nt.\end{aligned}\right\}$$

The variable part of the pressure on the tube side of the piston

$$=-h_0\frac{d\psi}{dt}.$$

The equation to the motion of the air in the mouth is therefore

$$\frac{Q}{c}\frac{d}{dt}\frac{d\psi}{dx}+\frac{d\psi}{dt}=0,$$

or, on integration,

$$\frac{Q}{c}\frac{d\psi}{dx}+\psi=0. \quad \ldots \ldots \ldots \ldots \quad (14)$$

This is the condition to be satisfied when $x=0$.

Substituting the values of ψ and $\frac{d\psi}{dx}$, we obtain

$$\cos 2\pi nt\left(A\frac{Q}{c}+B\right)+\beta\sin 2\pi nt=0,$$

which requires

$$A\frac{Q}{c}+B=0, \qquad \beta=0.$$

If there is a node at $x=-l$

$$A\cos kl+Bk\sin kl=0;$$

$$\therefore k\tan kl=-\frac{A}{B}=-\frac{c}{Q}. \quad \ldots \ldots \ldots \ldots \quad (15)$$

This equation gives the fundamental note of the tube closed at $x=-l$; but it must be observed that l is not the length of the tube, because the origin $x=0$ is not in the mouth. There is, however, nothing indeterminate in the equation, although the origin is to a certain extent arbitrary, for the values of c and l will change together so as to make the result for k approximately constant. This will appear more clearly when we come, in Part II., to calculate the actual value of c for different kinds of mouths. In the formation of (14) the pressure of the air on the positive side at a distance from the origin small against λ has been taken absolutely constant. Across such a loop surface no energy could be transmitted. In reality, of course, the pressure is variable on account of the spherical waves, and energy continually escapes from the tube and its vicinity. Although the pitch of the resonant note is not affected, it may be worth while to see what correction this involves.

We must, as before, consider the space in which the transition from plane to spherical waves is effected as small compared with λ. The potential in free space may be taken

$$\psi=\frac{A'}{r}\cos(kr+g-2\pi nt), \quad \ldots \ldots \ldots \ldots \quad (16)$$

expressing spherical waves diverging from the mouth of the pipe, which is the origin of r. The origin of x is still supposed to lie in the region of plane waves.

$$*4\pi r^2\frac{d\psi}{dr}=\text{rate of total flow across the surface of the sphere whose radius is } r$$

$$=-4\pi A'[\cos 2\pi nt\{\cos(kr+g)+kr\sin(kr+g)\}+\sin 2\pi nt\{\sin(kr+g)-kr\cos(kr+g)\}].$$

If the compression in the neighbourhood of the mouth is neglected, this must be the same as

$$Q\frac{d\psi}{dx=0}=QA\cos 2\pi nt.$$

Accordingly

$$AQ=-4\pi A'\{\cos(kr+g)+kr\sin(kr+g)\},$$
$$0=\sin(kr+g)-kr(\cos kr+g).$$

These equations express the connexion between the plane and spherical waves. From the second, $\tan(kr+g)=kr$, which shows that g is a small quantity of the order $(kr)^2$. From the first

$$A'=-\frac{AQ}{4\pi},$$

so that

$$\psi_r=-\frac{AQ}{4\pi r}\cos 2\pi nt-\frac{AQk}{4\pi}\sin 2\pi nt,$$

the terms of higher order being omitted.

Now within the space under consideration the air moves according to the same laws as electricity, and so

$$\frac{Q}{c}\frac{d\psi}{dx=0}=-\psi_{x=0}+\psi_r,$$

$$\frac{d\psi}{dx=0}=A\cos 2\pi nt,$$

$$\psi_{x=0}=B\cos 2\pi nt+\beta\sin 2\pi nt.$$

Therefore on substitution and equation of the coefficients of $\sin 2\pi nt$, $\cos 2\pi nt$, we obtain

$$\left.\begin{aligned}AQ\left(\frac{1}{c}+\frac{1}{4\pi r}\right)&=-B,\\ \beta&=-\frac{AQk}{4\pi}.\end{aligned}\right\}$$

When the mouth is not much contracted c is of the order of the radius of the mouth, and when there is contraction it is smaller still. In all cases therefore the term $\frac{1}{4\pi r}$ is very small compared to $\frac{1}{c}$; and we may put

$$\frac{AQ}{c}=-B,\quad \beta=-\frac{AQk}{4\pi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

* Throughout HELMHOLTZ's paper the mouth of the pipe is supposed to lie in an infinite plane, so that the diverging waves are hemispherical. The calculation of the value of c is thereby simplified. Except for this reason it seems better to consider the diverging waves completely spherical as a nearer approximation to the actual circumstances of organ-pipes, although the sphere could never be quite complete.

which agree nearly with the results of HELMHOLTZ. In his notation a quantity α is used defined by the equation

$$-\frac{A}{Bk}=\cot k\alpha,$$

so that

$$\cot k\alpha=\tan kl \text{ by } (15),$$

or

$$k(l+\alpha)=(2m+1)\frac{\pi}{2};$$

α may accordingly be considered as the correction to the length of the tube (measured, however, in our method only on the negative side of the origin), and will be given by

$$\cot k\alpha=-\frac{c}{kQ}.$$

The value of c will be investigated in Part II.

The original theory of open pipes makes the pressure absolutely constant at the mouth, which amounts to neglecting the inertia of the air outside. Thus, if the tube itself were full of air, and the external space of hydrogen, the correction to the length of the pipe might be neglected. The first investigation, in which no escape of energy is admitted, would apply if the pipe and a space round its mouth, large compared to the diameter, but small compared to the wave-length, were occupied by air in an atmosphere otherwise composed of incomparably lighter gas. These remarks are made by way of explanation, but for a complete discussion of the motion as determined by (13) and (17), I must refer to the paper of HELMHOLTZ.

Long Tube in connexion with a Reservoir.

It may sometimes happen that the length of a neck is too large compared to the quarter wave-length to allow the neglect of the compressibility of the air inside. A cylindrical neck may then be treated in the same way as the organ-pipe. The potential of plane waves inside the neck may, by what has been proved, be put into the form

$$\psi=A'\sin k(x-\alpha)\cos 2\pi nt;$$

if we neglect the escape of energy, which will not affect the pitch of the resonant note,

$$\frac{d\psi}{dt}=-2\pi nA'\sin k(x-\alpha)\sin 2\pi nt,$$

$$\frac{d\psi}{dx}=kA'\cos k(x-\alpha)\cos 2\pi nt,$$

where α is the correction for the outside end.

The rate of flow out of $S=Q\frac{d\psi}{dx}$.

Total flow $=Q\int\frac{d\psi}{dx}dt=kA'Q\cos kL\frac{\sin 2\pi nt}{2\pi n}$, the reduced length of the tube, including the corrections for both ends, being denoted by L. Thus rarifaction in S

$$=k\frac{A'Q\cos kL}{S}\frac{\sin 2\pi nt}{2\pi n}=\frac{1}{a^2}\frac{d\psi}{dt}=\frac{2\pi nA'\sin kL}{a^2}\sin 2\pi nt.$$

This is the condition to be satisfied at the inner end. It gives

$$\tan kL=\frac{a^2}{4\pi^2n^2}\frac{kQ}{S}=\frac{Q}{kS}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

When kL is small,

$$\tan kL=kL+\frac{1}{3}(kL)^3=\frac{Q}{kS};$$

$$\therefore\ k^2=\frac{Q}{LS}\left(1-\frac{1}{3}\frac{LQ}{S}\right),$$

$$n=\frac{a}{2\pi}\sqrt{\frac{Q}{LS}}\left(1-\frac{1}{6}\frac{LQ}{S}\right)=\frac{a}{2\pi}\sqrt{\frac{Q}{L(S+\frac{1}{3}LQ)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

In comparing this with (5), it is necessary to introduce the value of c, which is $\frac{Q}{L}$. (5) will accordingly give the same result as (19) if *one-third* of the contents of the neck be included in S. The first overtone, which is often produced by blowing in preference to the fundamental note, corresponds approximately to the length L of a tube open at both ends, modified to an extent which may be inferred from (18) by the finiteness of S.

The number of vibrations is given by

$$n=\frac{a}{2}\left(\frac{1}{L}+\frac{Q}{\pi^2S}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

[The application of (20) is rather limited, because, in order that the condensation within S may be uniform as has been supposed, the linear dimension of S must be considerably less than the quarter wave-length; while, on the other hand, the method of approximation by which (20) is obtained from (18) requires that S should be large in comparison with QL.

A slight modification of (18) is useful in finding the pitch of pipes which are cylindrical through most of their length, but at the closed end expand into a bulb S of no great capacity. The only change required is to understand by L the length of the pipe down to the place where the enlargement begins with a correction for the *outer* end. Or if L denote the length of the tube simply, we have

$$\tan k(L+\alpha)=\frac{Q}{kS}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20\,a)$$

and $\alpha=\frac{\pi}{4}$ R approximately.

If S be very small we may derive from (20 *a*)

$$n=\frac{a}{4\left(L+\alpha+\frac{S}{Q}\right)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20\,b)$$

In this form the interpretation is very simple, namely, that at the closed end the shape is of no consequence, and only the volume need be attended to. The air in this part of the pipe acts merely as a spring, its inertia not coming into play. A few measurements of this kind will be given in Part III.

The overtones of resonators which have not long necks are usually very high. Within the body of the reservoir a nodal surface must be formed, and the air on the further side vibrates as if it was contained in a completely closed vessel. We may form an idea of the character of these vibrations from the case of a sphere, which may be easily worked out from the equations given by Professor STOKES in his paper "On the Communication of Motion from a vibrating Sphere to a Gas"*. The most important vibration within a sphere is that which is expressed by the term of the first order in LAPLACE'S series, and consists of a swaying of the air from side to side like that which takes place in a doubly closed pipe. I find that for this vibration

$$\text{radius} : \text{wave-length} = {\cdot}3313,$$

so that the note is higher than that belonging to a doubly closed (or open) pipe of the length of the diameter of the sphere by about a musical fourth. We might realize this vibration experimentally by attaching to the sphere a neck of such length that it would by itself, when closed at one end, have the same resonant note as the sphere.

Lateral Openings.

In most wind instruments the gradations of pitch are attained by means of lateral openings, which may be closed at pleasure by the fingers or otherwise. The common crude theory supposes that a hole in the side of, say, a flute establishes so complete a communication between the interior and the surrounding atmosphere, that a loop or point of no condensation is produced immediately under it. It has long been known that this theory is inadequate, for it stands on the same level as the first approximation to the motion in an open pipe in which the inertia of the air outside the mouth is virtually neglected. Without going at length into this question, I will merely indicate how an improvement in the treatment of it may be made.

Let ψ_1, ψ_2 denote the velocity-potentials of the systems of plane waves on the two sides of the aperture, which we may suppose to be situated at the point $x=0$. Then with our previous notation the conditions evidently are that when $x=0$,

$$\left.\begin{aligned}&\psi_1=\psi_2\\&\frac{Q}{c}\left(\frac{d\psi_1}{dx}-\frac{d\psi_2}{dx}\right)+\psi=0,\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20\,c)$$

the escape of energy from the tube being neglected. These equations determine the connexion between the two systems of waves in any case that may arise, and the working out is simple. The results are of no particular interest, unless it be for a comparison with experimental measurements, which, so far as I am aware, have not hitherto been made.]

* Professor STOKES informs me that he had himself done this at the request of the Astronomer Royal.

PART II.

In order to complete the theory of resonators, it is necessary to determine the value of c, which occurs in all the results of Part I., for different forms of mouths. This we now proceed to do. Frequent use will be made of a principle which might be called that of minimum *vis viva*, and which it may be well to state clearly at the outset.

Imagine a portion of incompressible fluid at rest within a closed surface to be suddenly set in motion by an arbitrary normal velocity impressed on the surface, then the actual motion assumed by the fluid will have less *vis viva* than any other motion consistent with continuity and with the boundary conditions*.

If u, v, w be the component velocities, and ρ the density at any point,

$$\textit{vis viva} = \tfrac{1}{2}\iiint \rho(u^2+v^2+w^2)\,dx\,dy\,dz,$$

the integration extending over the volume considered. The minimum *vis viva* corresponding to prescribed boundary conditions depends of course on ρ; but if in any specified case we conceive the value of ρ in some places diminished and nowhere increased, we may assert that the minimum *vis viva* is *less* than before; for there will be a decrease if u, v, w remain unaltered, and therefore, *à fortiori*, when they have their actual values as determined by the minimum property. Conversely, an increase in ρ will necessarily raise the value of the minimum *vis viva*. The introduction of a rigid obstacle into a stream will always cause an increase of *vis viva*; for the new motion is one that might have existed before consistently with continuity, the fluid displaced by the obstacle remaining at rest. Any kind of obstruction in the air-passages of a musical instrument will therefore be accompanied by a fall of the note in the musical scale.

Long Tubes.

The simplest case that can be considered consists of an opening in the form of a cylindrical tube, so long in proportion to its diameter that the corrections for the ends may be neglected. If the length be L and area of section σ, the electrical resistance is $\frac{L}{\sigma}$, and

$$c=\frac{\sigma}{L}. \qquad (21)$$

For a circular cylinder of radius R

$$c=\frac{\pi R^2}{L}. \qquad (22)$$

Simple Apertures.

The next in order of simplicity is probably the case treated by HELMHOLTZ, where the opening consists of a simple hole in the side of the reservoir, considered as indefinitely thin and approximately plane in the neighbourhood of the opening. The motion of the

* THOMSON and TAIT's 'Natural Philosophy,' p. 230.

fluid in the plane of the opening is by the symmetry normal, and therefore the velocity-potential is constant over the opening itself. Over the remainder of the plane in which the opening lies the normal velocity is of course zero, so that ϕ may be regarded as the potential of matter distributed over the opening only. If the there constant value of the potential be called ϕ_1, the electrical resistance for *one side only* is

$$\phi_1 \div \iint \frac{d\phi}{dn} d\sigma,$$

the integration going over the area of the opening.

Now

$$\iint \frac{d\phi}{dn} d\sigma = 2\pi \times \text{ the whole quantity of matter;}$$

so that if we call M the quantity necessary to produce the unit potential,

$$\text{resistance for one side} = \frac{1}{2\pi M}.$$

Accordingly

$$c = \pi M. \quad \ldots \quad (23)$$

In electrical language M is the *capacity* of a conducting lamina of the shape of the hole when situated in an open space.

For a circular hole $M = \frac{2R}{\pi}$, and therefore

$$c = 2R. \quad \ldots \quad (24)$$

When the hole is an ellipse of eccentricity e and semimajor axis R,

$$c = \frac{\pi R}{F(e)}, \quad \ldots \quad (25)$$

where F is the symbol of the complete elliptic function of the first order. Results equivalent to (23), (24), and (25) are given by HELMHOLTZ.

When the eccentricity is but small, the value of c depends sensibly on the area (σ) of the orifice only. As far as the square of e,

$$F(e) = \frac{\pi}{2}(1 + \tfrac{1}{4}e^2),$$

$$\sigma = \pi R^2 \sqrt{1 - e^2} = \pi R^2 (1 - \tfrac{1}{2}e^2),$$

$$R = \sqrt{\frac{\sigma}{\pi}}(1 + \tfrac{1}{4}e^2);$$

$$\therefore \ c = \pi \sqrt{\frac{\sigma}{\pi}} \div \frac{\pi}{2} = 2\sqrt{\frac{\sigma}{\pi}}, \quad \ldots \quad (26)$$

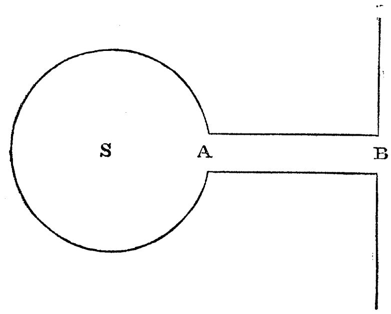
the fourth power of e being neglected—a formula which may be applied without sensible error to any orifice of an approximately circular form. In fact for a given area the circle is the figure which gives a minimum value to c, and in the neighbourhood of the minimum the variation is slow.

Next, consider the case of two circular orifices. If sufficiently far apart they act independently of each other, and the value of c for the pair is the simple sum of the separate values, as may be seen either from the law of multiple arcs by considering c as the electric *conductivity* between the outside and inside of the reservoir, or from the interpretation of M in (23). The first method applies to any kind of openings with or without necks. As the two circles (which for precision of statement we may suppose equal) approach one another, the value of c diminishes steadily until they touch. The change in the character of the motion may be best followed by considering the plane of symmetry which bisects at right angles the line joining the two centres, and which may be regarded as a rigid plane precluding normal motion. Fixing our attention on half the motion only, we recognize the plane as an obstacle continually advancing, and at each step more and more obstructing the passage of fluid through the circular opening. After the circles come into contact this process cannot be carried further; but we may infer that, as they amalgamate and shape themselves into a single circle (the total area remaining all the while constant), the value of c still continues to diminish till it approaches its minimum value, which is less than at the commencement in the ratio of $\sqrt{2}:2$ or $1:\sqrt{2}$. There are very few forms of opening indeed for which the exact calculation of M or c can be effected. We must for the present be content with the formula (26) as applying to nearly circular openings, and with the knowledge that the more elongated or broken up the opening, the greater is c compared to σ. In the case of similar orifices or systems of orifices c varies as the linear dimension.

Cylindrical Necks.

Most resonators used in practice have necks of greater or less length, and even where there is nothing that would be called a neck, the thickness of the side of the reservoir could not always be neglected. For simplicity we shall take the case of circular cylinders whose inner ends lie on an approximately plane part of the side of the vessel, and whose outer ends are also supposed to lie in an infinite plane, or at least a plane whose dimensions are considerable compared to the diameter of the cylinder. Even under this form the problem does not seem capable of exact solution; but we shall be able to fix two slightly differing quantities between which the true value of c must lie, and which determine it with an accuracy more than sufficient for acoustical purposes. The object is to find the *vis viva* in terms of the rate of flow. Now, according to the principle stated at the beginning of Part II., we shall obtain too small a *vis viva* if at the ends A and B of the tube we imagine infinitely thin laminæ of fluid of infinitely small density. We may be led still more distinctly perhaps to the same result by supposing, in the electrical analogue, thin disks of perfectly conducting matter at the ends of the tube, whereby the effective resistance must plainly be lessened. The action of the disks is to produce uniform

Fig. 4.



potential over the ends, and the solution of the modified problem is obvious. Outside the tube the question is the same as for a simple circular hole in an infinite plane, and inside the tube the same as if the tube were indefinitely long.

Accordingly

$$\text{resistance}=\frac{L}{\pi R^2}+\frac{1}{2R}=\frac{1}{\pi R^2}\left(L+\frac{\pi}{2}R\right). \quad \text{. . . . . . .} \quad (27)$$

The correction to the length is therefore $\frac{\pi}{2}R$, that is, $\frac{\pi}{4}R$ for each end,

$$c=\frac{\pi R^2}{L+\frac{\pi}{2}R}. \quad \text{. . . . .} \quad (28)$$

HELMHOLTZ, in considering the case of an organ-pipe, arrives at a similar conclusion,—that the correction to the length (α) is approximately $\frac{\pi}{4}R$. His method is very different from the above, and much less simple. He begins by investigating certain forms of mouths for which the exact solution is possible, and then, by assigning suitable values to arbitrary constants, identifies one of them with a true cylinder, the agreement being shown to be everywhere very close. Since the curve substituted for the generating line of the cylinder lies entirely outside it, HELMHOLTZ infers that the correction to the length thus obtained is too small.

If, at the ends of the tube, instead of layers of matter of no density, we imagine rigid pistons of no sensible thickness, we shall obtain a motion whose *vis viva* is necessarily *greater* than that of the real motion; for the motion with the pistons might take place without them consistently with continuity. Inside the tube the character of the motion is the same as before, but for the outside we require the solution of a fresh problem:—To determine the motion of an infinite fluid bounded by an infinite plane, the normal velocity over a circular area of the plane being a given constant, and over the rest of the plane zero. The potential may still be regarded as due to matter confined to the circle, but is no longer constant over its area; but the density of matter at any point, being proportional to $\frac{d\phi}{dn}$ or to the normal velocity, is constant.

The *vis viva* of the motion

$$=\tfrac{1}{2}\iint\phi\frac{d\phi}{dn}d\sigma=\tfrac{1}{2}\frac{d\phi}{dn}\iint\phi d\sigma,$$

the integration going over the area of the circle.

The rate of total flow through the plane

$$=\iint\frac{d\phi}{dn}d\sigma=\pi R^2\frac{d\phi}{dn};$$

$$\therefore\frac{2\ \textit{vis viva}}{(\text{rate of flow})^2}=\frac{\iint\phi d\sigma}{\pi^2R^4\frac{d\phi}{dn}}. \quad \text{. . . . .} \quad (29)$$

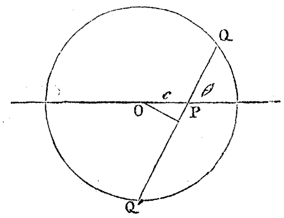
We proceed to investigate the value of $\iint\phi d\sigma$, which is the *potential on itself* of a circular disk of unit density.

Potential on itself of a uniform circular disk.

r denoting the distance between any two points on the disk, the quantity to be evaluated is expressed by

$$\iint d\sigma \iint \frac{d\sigma'}{r}.$$

Fig. 5.



The first step is to find the potential at any point P, or $\iint\frac{d\sigma'}{r}$. Taking this point as an origin of polar coordinates, we have

$$\text{potential} = \iint\frac{d\sigma'}{r} = \iint\frac{rdrd\theta}{r} = \int rd\theta = \int(\text{PQ}+\text{PQ}')d\theta.$$

Now from the figure

$$\tfrac{1}{4}(\text{QQ}')^2 = \text{R}^2 - c^2\sin^2\theta,$$

where c is the distance of the point P from the centre of the circle whose radius is R. Thus potential at P

$$=2\text{R}\int_0^\pi\sqrt{1-\frac{c^2}{\text{R}^2}\sin^2\theta}\,d\theta = 4\text{R}\int_0^{\frac{\pi}{2}}\sqrt{1-\frac{c^2}{\text{R}^2}\sin^2\theta}\,d\theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Hence potential of disk on itself

$$=4\pi\text{R}^3\int_0^1 dx\int_0^{\frac{\pi}{2}}\sqrt{1-x\sin^2\theta}\,d\theta,$$

if for the sake of brevity we put $\frac{c^2}{\text{R}^2}=x$.

In performing first the integration with respect to θ we come upon elliptic functions, but they may be avoided by changing the order of integration.

$$\int_0^1 dx\sqrt{1-x\sin^2\theta} = \left\{-\frac{2}{3\sin^2\theta}(1-x\sin^2\theta)^{\frac{3}{2}}\right\}_0^1$$

$$=\frac{2}{3\sin^2\theta}(1-\cos^3\theta) = \frac{2}{3}\frac{1}{1+\cos\theta}+\frac{2}{3}\cos\theta;$$

∴ potential on itself

$$=\frac{8}{3}\pi\text{R}^3\int_0^{\frac{\pi}{2}}d\theta\left\{\frac{1}{1+\cos\theta}+\cos\theta\right\} = \frac{8}{3}\pi\text{R}^3\{1+1\} = \frac{16}{3}\pi\text{R}^3. \quad . \quad . \quad . \quad . \quad . \quad (31)$$

This, therefore, is the value of $\iint\phi d\sigma$ when the density is supposed equal to unity. The corresponding value of

$$\frac{d\phi}{dn}=2\pi,$$

and so from (29)

$$\frac{2\ vis\ viva}{(\text{rate of flow})^2}=\frac{8}{3\pi^2 R}. \qquad (32)$$

This is for the space outside one end. For the whole tube and both ends

$$\frac{2\ vis\ viva}{(\text{rate of flow})^2}=\frac{L}{\pi R^2}+\frac{16}{3\pi^2 R}. \qquad (33)$$

Whatever, then, may be the ratio of L : R, the electrical resistance to the passage in question or $\frac{1}{c}$ is limited by

$$\left.\begin{aligned}\frac{1}{c}&>\frac{L}{\pi R^2}+\frac{1}{2R}\\ &<\frac{L}{\pi R^2}+\frac{16}{3\pi^2 R}\end{aligned}\right\} \qquad (34)$$

In practical application it is sometimes convenient to use the quantity α or correction to the length. In terms of α (34) becomes

$$\left.\begin{aligned}\alpha&>\frac{\pi}{2}R\\ &<\frac{16}{3\pi}R,\end{aligned}\right\}$$

or in decimals,

$$\left.\begin{aligned}\alpha&>(1{\cdot}571R=2\times{\cdot}785R)\\ \alpha&<(1{\cdot}697R=2\times{\cdot}849R)\end{aligned}\right\}. \qquad (34')$$

The corrections for *both* ends is the thing here denoted by α. Of course for one end it is only necessary to take the half*.

I do not suppose that any experiments hitherto made with organ-pipes could discriminate with certainty between the two values of α in (34′). If we adopt the mean provisionally, we may be sure that we are not wrong by so much as ·032 R for each end.

Our upper limit to the value of α expressed in (34′) was found by considering the hypothetical case of a uniform velocity over the section of the mouth, and we fully determined the non-rotational motion both for the inside and for the outside of the tube. Of course the velocity is not really uniform at the mouth; it is, indeed, infinite at the edge. If we could solve the problem for the inside and outside when the velocity (normal) at the mouth is of the form $a+br^2$, we should with a suitable value of $b:a$ get a much better approximation to the true *vis viva*. The problem for the outside may be solved, but for the inside it seems far from easy. It is possible, however, that we may

* Though not immediately connected with our present subject, it may be worth notice that if at the centre of the tube, or anywhere else, the velocity be constrained (by a piston) to be constant across the section, as it would approximately be if the tube were very long, without a piston, the limiting inequalities (34) still hold good. For large values of L the two cases do not sensibly differ, but for small values of L compared to R the true solution of the original problem tends to coincide with the lower limit, and of the modified (central piston) problem with the higher.

be able to find some motion for the inside satisfying the boundary conditions and the equation of continuity, which, though of a rotational character, shall yet make the whole *vis viva* for the inside and outside together less than that previously obtained. At the same time this *vis viva* is by THOMSON'S law necessarily greater than the one we seek.

Motion in a finite cylindrical tube, the axial velocity at the plane ends ($x=0$ and $x=l$) *being*

$$u=u_0+\chi(r), \qquad (35)$$

where

$$\int_0^1 r\chi(r)dr=0, \qquad (36)$$

r being the transverse coordinate, and the radius of the cylinder being put equal to 1.

If u, v be the component velocities, the continuity equation is

$$\frac{du}{dx}+\frac{1}{r}\frac{d(rv)}{dr}=0, \qquad (37)$$

whence

$$\left.\begin{aligned} ru&=\frac{d\psi}{dr},\\ rv&=-\frac{d\psi}{dx},\end{aligned}\right\} \qquad (37')$$

where ψ is arbitrary so far as (37) is concerned.

Take

$$\psi=u_0\frac{r^2}{2}+\phi(x)\int_0^r r\chi(r)dr,$$

so that

$$\left.\begin{aligned} u&=u_0+\phi(x)\chi(r),\\ v&=-\phi'(x)\frac{1}{r}\int_0^r r\chi r dr.\end{aligned}\right\} \qquad (38)$$

It is clear from (38) that if

$$\phi(0)=\phi(l)=1, \qquad (39)$$

$$u_{x=0}=u_0+\chi r=u_{x=l}$$

$$v_{r=1}=0 \text{ for all values of } x.$$

Thus (38) satisfies the boundary conditions including (35), and ϕ is still arbitrary, except in so far as it is limited by (39).

In order to obtain an expression for the *vis viva*, we must integrate u^2+v^2 over the volume of the cylinder.

$$\left.\begin{aligned} \text{Twice } \textit{vis viva}&=u_0^2\pi l+2u_0\int_0^l \phi(x)dx\int_0^1 \chi(r)2\pi r dr\\ &+\int_0^l \overline{\phi x}|^2 dx\int_0^1 2\pi\overline{\chi(r)}|^2 r dr\\ &+\int_0^l \overline{\phi' x}|^2 dx\int_0^1 \frac{2\pi dr}{r}\left\{\int_0^r r\chi(r)dr\right\}^2.\end{aligned}\right\} \qquad (40)$$

The second term vanishes in virtue of (36), and we may write

$$\text{Twice } vis\ viva = u_0^2\pi l + \int_0^l (Ay^2 + By'^2)dx, \quad \text{(40')}$$

where A and B are known quantities depending on χ, and $y=\phi(x)$ is so far an arbitrary function, which we shall determine so as to make the *vis viva* a minimum.

By the method of variations

$$y = C\varepsilon^{-\sqrt{\frac{A}{B}}x} + C'\varepsilon^{+\sqrt{\frac{A}{B}}x}; \quad (41)$$

and in order to satisfy (39),

$$\left.\begin{aligned} 1 &= C + C', \\ 1 &= C\varepsilon^{-\sqrt{\frac{A}{B}}l} + C'\varepsilon^{+\sqrt{\frac{A}{B}}l} \end{aligned}\right\} \quad (42)$$

(41) and (42) completely determine y as a function of x, and when this value of y is used in (40) the *vis viva* is less than with any other form of y. On substitution in (40'),

$$\text{Twice } vis\ viva = u_0^2\pi l + 2\sqrt{AB}\,\frac{1-\varepsilon^{-2\sqrt{\frac{A}{B}}l}}{1+\varepsilon^{-2\sqrt{\frac{A}{B}}l}}. \quad (43)$$

The *vis viva* expressed in (43) is less than any other which can be derived from the equation (38); but it is not the least possible, as may be seen by substituting the value of ψ in the stream-line equation

$$\frac{d^2\psi}{dx^2} - \frac{1}{r}\frac{d\psi}{dr} + \frac{d^2\psi}{dr^2} = 0,$$

which will be found to be *not* satisfied.

The next step is to introduce special forms of χ. Thus let

$$u_{x=0} = 1 + \mu r^2.$$

Then

$$u_0 = 1 + \tfrac{1}{2}\mu,$$

$$\chi = \mu(-\tfrac{1}{2} + r^2).$$

Accordingly

$$A = \frac{\pi\mu^2}{12}, \qquad B = \frac{1}{16}\cdot\frac{\pi\mu^2}{12}, \qquad \sqrt{AB} = \frac{\pi\mu^2}{48}; \qquad \sqrt{\frac{A}{B}} = 4,$$

and (43) becomes

$$2\ vis\ viva = \pi l(1+\tfrac{1}{2}\mu)^2 + \frac{\pi\mu^2}{24}\,\frac{1-\varepsilon^{-8l}}{1+\varepsilon^{-8l}}. \quad (44)$$

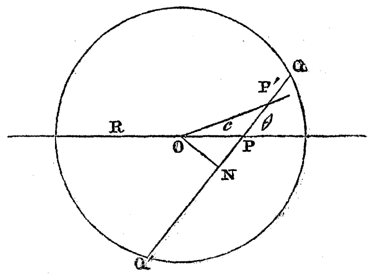
We have in (44) the *vis viva* of a motion within a circular cylinder which satisfies the continuity equation, and which makes over the plane ends

$$u = 1 + \mu r^2.$$

If $\mu=0$ we fall back on the simple case considered before; and this is the value of μ for which the *vis viva* in (44) is a minimum compared to the rate of flow $(1+\frac{1}{2}\mu)$. But for the part outside the cylinder the *vis viva* is, as we may anticipate, least when μ has some finite value; so that when we consider the motion as a whole it will be a finite value of μ that gives the least *vis viva*.

The *vis viva* of the motion outside the ends is to be found by the same method as before, the first step being to determine the potential at any point of a circular disk whose density $=\mu r^2$;

Fig. 6.



$$\text{potential at P}=\iint\frac{\varrho d\varrho d\theta}{\varrho}\mu \text{OP}'^2,$$

where

$$\text{OP}'^2=c^2+\varrho^2+2c\varrho\cos\theta;$$

$$\therefore\ \text{potential at P}=\int\mu d\theta\left\{c^2\varrho+\frac{\varrho^3}{3}+c\varrho^2\cos\theta\right\};$$

or if previously to integration with respect to θ we add together the elements from Q to Q′,

$$=\mu\int_0^\pi d\theta(\text{PQ}+\text{PQ}')\left\{c^2+\frac{\text{PQ}^2+\text{PQ}'^2-\text{PQ}\,.\,\text{PQ}'}{3}+c\cos\theta(\text{PQ}-\text{PQ}')\right\}.$$

Now

$$\text{PQ}+\text{PQ}'=2\sqrt{\text{R}^2-c^2\sin^2\theta},$$

$$\text{PQ}-\text{PQ}'=-2\text{PN}=-2c\cos\theta,$$

$$\text{PQ}\,.\,\text{PQ}'=\text{R}^2-c^2.$$

Thus potential at $\text{P}=\frac{4\text{R}^3}{3}\mu\int_0^{\frac{\pi}{2}}d\theta\sqrt{1-c^2\sin^2\theta}(1+2c^2\sin^2\theta)$, c being written for $c\div\text{R}$.

To this must be added the potential for a uniform disk found previously, and the result must be multiplied by the compound density and integrated again over the area, the order of integration being changed as before so as to take first the integration with respect to c. In this way elliptic functions are avoided; but the process is too long to be given here, particularly as it presents no difficulty. The result is that the potential on itself of a disk whose density

$$=1+\mu\frac{r^2}{\text{R}^2}$$

is expressed by

$$\frac{16\pi\text{R}^3}{3}(1+\tfrac{14}{15}\mu+\tfrac{5}{21}\mu^2)^* \quad\ldots\ldots\ldots\ldots \quad (45)$$

* [Mr. Clerk Maxwell has pointed out a process by which this result may be obtained much more simply. Begin by finding the potential at the *edge* of the disk whose density is $1+\mu r^2$. Taking polar coordinates (ρ, θ), the pole being at the edge, we have

$$r^2=\rho^2+a^2-2a\rho\cos\theta$$

and

$$\text{V}=\iint\{1+\mu(\rho^2+a^2-2a\rho\cos\theta)\}d\theta d\rho,$$

the limits of ρ being 0 and $2a\cos\theta$, and those of θ being $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$. We get at once

$$\text{V}=4a+\tfrac{20}{9}\mu a^3.$$

Thus if for brevity we put $R=1$, we may express the *vis viva* of the whole motion (both extremities included) by

$$2 \textit{ vis viva}=\pi l(1+\tfrac{1}{2}\mu)^2+\frac{\pi\mu^2}{24}\frac{1-\varepsilon^{-8l}}{1+\varepsilon^{-8l}}+\frac{16}{3}\left(1+\frac{14}{15}\mu+\frac{5}{21}\mu^2\right),$$

which corresponds to the rate of flow $\pi u_0=\pi(1+\frac{1}{2}\mu)$.

Thus

$$\frac{2 \textit{ vis viva}}{(\text{rate of flow})^2}=\frac{l}{\pi}+\frac{1}{3\pi}\frac{\frac{\Lambda}{8}\mu^2+\frac{16}{\pi}\left(1+\frac{14}{15}\mu+\frac{5}{21}\mu^2\right)}{(1+\frac{1}{2}\mu)^2}, \quad . \quad . \quad . \quad . \quad . \quad (46)$$

where $\Lambda=\frac{1-\varepsilon^{-8l}}{1+\varepsilon^{-8l}}$.

The second fraction on the right of (46) is next to be made a minimum by variation of μ. Putting it equal to z and multiplying up, we get the following quadratic in μ:—

$$\mu^2\left\{\frac{\Lambda}{8}+\frac{5\cdot 16}{21\pi}-\frac{z}{4}\right\}+2\mu\left\{\frac{16\cdot 7}{15\pi}-\frac{z}{2}\right\}+\frac{16}{\pi}-z=0.$$

The smallest value of z consistent with a real value of μ is therefore given by

$$\left(\frac{16\cdot 7}{15\pi}-\frac{z}{2}\right)^2-\left(\frac{16}{\pi}-z\right)\left(\frac{\Lambda}{8}+\frac{5\cdot 16}{21\pi}-\frac{z}{4}\right)=0$$

$$z=\frac{2\Lambda+\frac{8192}{1575\pi}}{\frac{\Lambda\pi}{8}+\frac{12}{35}}=\frac{2\Lambda+1\cdot 6556}{\cdot 3927\Lambda+\cdot 3429}=\frac{3\cdot 6556-\cdot 3444\varepsilon^{-8l}}{\cdot 7356-\cdot 0498\varepsilon^{-8l}}.$$

Thus

$$\frac{2 \textit{ vis viva}}{(\text{rate of flow})^2}=\frac{l}{\pi}+\frac{1}{3\pi}\,\frac{3\cdot 6556-\cdot 3444\varepsilon^{-8l}}{\cdot 7356-\cdot 0498\varepsilon^{-8l}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

This gives an upper limit to $\frac{1}{c}$. In terms of α (including both ends)

$$\alpha<2\cdot 305R\,\frac{10\cdot 615-\varepsilon^{-8\frac{l}{R}}}{14\cdot 771-\varepsilon^{-8\frac{l}{R}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47')$$

From (47′) we see that the limit for α is smallest when $l=0$, and gradually increases with l.

Now let us cut off a strip of breadth da from the edge of the disk, whose mass is accordingly

$$2\pi a(1+\mu a^2)da.$$

The work done in carrying this strip off to infinity is

$$2\pi a da(1+\mu a^2)(4a+\tfrac{20}{9}\mu a^3).$$

If we gradually pare the disk down to nothing and carry all the parings to infinity, we find for the total work by integrating with respect to a from 0 to R,

$$\frac{8\pi R^3}{3}\left(1+\frac{14}{15}\mu+\frac{5}{21}\mu^2\right),$$

μ being written for μR^2. This is, as it should be, the half of the expression in the text.]

When $l=\infty$, it becomes

$$1{\cdot}6565\,\mathrm{R}=2\times{\cdot}8282\,\mathrm{R}.$$

Thus the correction for one end of an infinite tube is limited by

$$\left.\begin{array}{l}\alpha>{\cdot}785\,\mathrm{R}\\ \quad<{\cdot}8282\,\mathrm{R}\end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

When l is not infinitely great the upper limit may be calculated from (47′), the lower limit remaining as before; but it is only for quite small values of l that the exponential terms in (47′) are sensible. It is to be remarked that the *real* value of α is least when $l=0$, and gradually increases to its limit when $l=\infty$. For consider the actual motion for any finite value of l. The *vis viva* of the motion going on in any middle piece of the tube is greater than corresponds merely to the length. If the piece therefore be removed and the ends brought together, the same motion may be supposed to continue without violation of continuity, and the *vis viva* will be more diminished than corresponds to the length of the piece cut out. *A fortiori* will this be true of the real motion which would exist in the shortened tube. Thus α steadily decreases as the tube is shortened until when $l=0$ it coincides with the lower limit $\frac{\pi}{4}\mathrm{R}$.

In practice the outer end of a rather long tube-like neck cannot be said generally to end in an infinite plane, as is supposed in the above calculation. On the contrary, there could ordinarily be a certain flow back round the edge of tube, the effect of which must be sensibly to diminish α. It would be interesting to know the exact value of α for an infinite tube projecting into unlimited space free from obstructing bodies, the thickness of the cylindrical tube being regarded as vanishingly small. HELMHOLTZ has solved what may be called the corresponding problem in two dimensions; but the difficulty in the two cases seems to be of quite a different kind. Fortunately our ignorance on this point is not of much consequence for acoustical purposes, because when the necks are short the hypothesis of the infinite plane agrees nearly with the fact, and when the necks are long the correction to the length is itself of subordinate importance.

Nearly Cylindrical Tubes of Revolution.

The non-rotational flow of a liquid in a tube of revolution or of electricity in a similar solid conductor can only in a few cases be exactly determined. It may therefore be of service to obtain formulæ fixing certain limits between which the *vis viva* or resistance must lie. First, considering the case of electricity (for greater simplicity of expression), let us conceive an indefinite number of infinitely thin but at the same time perfectly conducting planes to be introduced perpendicular to the axis. Along these the potential is necessarily constant, and it is clear that their presence must *lower* the resistance of the conductor in question. Now at the point x (axial coordinate) let the radius of the conductor be y, so that its section is πy^2. The resistance between two of the above-mentioned planes which are close to one another and to the point x will be in the limit

$dx \div \pi y^2$, if dx be the distance between the planes, the resistance of the unit cube being unity. Thus resistance

$$=\int \frac{dx}{\pi y^2} * \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Upper Limit.

Secondly, we know that in the case of a liquid the true *vis viva* is less than that of any other motion which satisfies the boundary conditions and the equation of continuity. Now u, v being the axial and transverse velocities, it will always be possible so to determine v as to satisfy the conditions if we assume $u=$ constant over the section, and therefore

$$u=\frac{u_0}{\pi y^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

This may be seen by imagining rigid pistons introduced perpendicular to the axis. To determine v it is convenient to use the function ψ, which is related to u and v according to the equations (37),

$$ru=\frac{d\psi}{dr}, \qquad rv=-\frac{d\psi}{dx}.$$

These forms for u and v secure the fulfilment of the continuity equation. Since

$$u=\frac{u_0}{\pi y^2}, \qquad \frac{d\psi}{dr}=r\frac{u_0}{\pi y^2},$$

$$\psi=\psi_0(x)+\frac{u_0}{2\pi y^2}r^2,$$

and therefore

$$v=-\frac{\psi_0(x)}{r}-\frac{u_0}{2\pi}r\frac{d}{dx}\left(\frac{1}{y^2}\right).$$

But since v cannot be infinite on the axis, but must, on the contrary, be zero,

$$\psi_0(x)=0,$$

and we have

$$\left.\begin{aligned} u&=\frac{u_0}{\pi y^2}, \\ v&=-\frac{u_0}{2\pi}r\frac{d}{dx}\left(\frac{1}{y^2}\right) \end{aligned}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

From the manner in which these were obtained, they must satisfy the condition of

* [It is easy to show formally that no error can arise from neglecting the effect of the curved rim. Imagine the planes at x and $x+dx$ extended, and the curves in which they cut the surface of the conductor projected by lines parallel to the axis. In this way a cylinder is formed which contains the whole surface between x and $x+dx$, and another cylinder which is entirely contained by the surface. The small cylinder may be obtained by supposing part of the matter not to conduct, and therefore gives too great a resistance. On the other hand, the real solid may be obtained from the large cylinder by the same process. The resistance of the slice lies accordingly between those of the two cylinders which are themselves equal in the limit. Hence, on the whole, the parts neglected vanish compared to those retained.]

giving no normal motion at the surface of the tube. That this is actually the case may be easily verified *à posteriori*, but it is scarcely necessary for our purpose to do so. To find the *vis viva*,

$$\int_0^y u^2 2\pi r dr = \frac{u_0^2}{\pi y^2},$$

$$\iint u^2 2\pi r dr dx = \frac{u_0^2}{\pi}\int \frac{1}{y^2}dx,$$

$$\int v^2 2\pi r dr = \frac{u_0^2 y^4}{8\pi}\left[\frac{d}{dx}\left(\frac{1}{y^2}\right)\right]^2,$$

$$\iint v^2 2\pi r dr dx = \frac{u_0^2}{8\pi}\int y^4\left[\frac{d}{dx}\left(\frac{1}{y^2}\right)\right]^2 dx = \frac{u_0^2}{2\pi}\int \frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 dx.$$

Thus $$vis\ viva = \frac{u_0^2}{2\pi}\int \frac{1}{y^2}\left\{1+\tfrac{1}{2}\left(\frac{dy}{dx}\right)^2\right\}dx.$$

The total flow across any section is $\pi y^2 u = u_0$.

Therefore

$$\frac{2\ vis\ viva}{(\text{rate of flow})^2} = \frac{1}{\pi}\int \frac{1}{y^2}\left\{1+\frac{1}{2}\left(\frac{dy}{dx}\right)^2\right\}dx. \quad \quad (52)$$

This is the quantity which gives an upper limit to the resistance. The first term, which corresponds to the component u of the velocity, is the same as that previously obtained for the lower limit, as might have been foreseen. The difference between the two, which gives the utmost error involved in taking either of them as the true value, is

$$\frac{1}{2\pi}\int \frac{1}{y^2}\left(\frac{dy}{dx}\right)^2 dx.$$

In a nearly cylindrical tube $\frac{dy}{dx}$ is a small quantity, and so the result found by this method is closely approximate. It is not necessary that the section of the tube should be nearly constant, but only that it should vary slowly. The success of the approximation in this and similar cases depends in great measure on the fact that the quantity to be estimated is a minimum. Any reasonable approximation to the real motion will give a *vis viva* very near the minimum, according to the principles of the differential calculus.

Application to straight tube of revolution whose end lies on two infinite planes.

For the lower limit to the resistance we have

$$\frac{1}{\pi}\int \frac{dx}{y^2} + \frac{1}{4R_1} + \frac{1}{4R_2},$$

R_1, R_2 being the radii at the ends, and for the higher limit

$$\frac{1}{\pi}\int \frac{1}{y^2}\left\{1+\tfrac{1}{2}\left(\frac{dy}{dx}\right)^2\right\}dx + \frac{8}{3\pi^2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right).$$

The first expression is obtained by supposing infinitely thin but perfectly conducting planes perpendicular to the axis to be introduced from the ends of the tube inwards, while in the second the conducting planes in the electical interpretation are replaced by pistons in the hydrodynamical analogue. For example, let the tube be part of a cone of semivertical angle θ.

The lower limit is

$$\frac{1}{\pi \tan \theta}\left(\frac{1}{R_1} \sim \frac{1}{R_2}\right)+\frac{1}{4}\left(\frac{1}{R_1}+\frac{1}{R_2}\right),$$

and the higher

$$\frac{1+\tan^2\theta}{\pi \tan \theta}\left(\frac{1}{R_1} \sim \frac{1}{R_2}\right)+\frac{8}{3\pi^2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right).$$

Tubes nearly straight and cylindrical but not necessarily of revolution.

Taking the axis of x in the direction of the length, we readily obtain by the same process as before a *lower* limit to the resistance

$$\int \frac{dx}{\sigma}, \quad \ldots \ldots \ldots \ldots \quad (53)$$

where σ denotes the section of the tube by a plane perpendicular to the axis at the point x, an expression which has long been known and is sometimes given as rigorous. The conductor (for I am now referring to the electrical interpretation) is conceived to be divided into elementary slices by planes perpendicular to the axis, and the resistance of any slice is calculated as if its faces were at constant potentials, which is of course not the case. In fact it is meaningless to talk of the resistance of a limited solid at all, unless with the understanding that certain parts of its surface are at constant potentials, while other parts are bounded by non-conductors. Thus, when the resistance of a cube is spoken of, it is tacitly assumed that two of the opposite faces are at constant potentials, and that the other four faces permit no escape of electricity across them. In some cases of unlimited conductors, for instance one we have already contemplated—an infinite solid almost divided into two separate parts by an infinite insulating plane with a hole in it—it is allowable to speak of the resistance without specifying what particular surfaces are regarded as equipotential; for at a sufficient distance from the opening on either side the potential is constant, and any surface no part of which approaches the opening is approximately equipotential. After this explanation of the exact significance of (53), we may advantageously modify it into a form convenient for practical use.

The section of the tube at n different points of its length l is obtained by observing the length λ of a mercury thread which is caused to traverse the tube. Replacing the integration by a summation denoted by the symbol Σ, we arrive at the formula

$$\text{resistance}=\frac{l^2}{n^2 V} \Sigma\lambda\, \Sigma \frac{1}{\lambda}, \quad \ldots \ldots \ldots \ldots \quad (54)$$

which was used by Dr. MATTHIESSEN in his investigation of the mercury unit of electrical resistance, and was the subject of some controversy*. It is perfectly correct in the sense that when the number of observations is increased without limit it coincides with (53), *itself, however, only an approximation to the magnitude sought.* The extension of our second method (for the higher limit) to tube not of revolution would require the general solution of the potential problem in two dimensions. It may be inferred that the difference between the two limits is of the order of the square of the inclination of the tangent plane to the axis, and is therefore very small when the section of the tube alters but slowly.

Tubes not nearly straight.

In applying (53) to such cases, we are at liberty to take any straight line we please as axis; but if the tube is much bent, even though its cross section remain nearly constant, the approximation will cease to be good. This is evident, because the planes of constant potential must soon become very oblique, and the section σ used in the formula much greater than the really effective section of the tube. To meet this difficulty a modification in the formula is necessary. Instead of taking the artificial planes of equal potential all perpendicular to a straight line, we will now take them normal to a curve which may have double curvature, and which should run, as it were, along the middle of the tube. Consecutive planes intersect in a straight line passing through the centre of curvature of the "axis" and perpendicular to its plane.

The resistance between two neighbouring equipotential planes is in the limit

$$\delta\theta \div \iint \frac{d\sigma}{r},$$

where $\delta\theta$ is the angle between the planes, and r is the distance of any element $d\sigma$ of the

* See SABINE's 'Electric Telegraph,' p. 329. To prove (54), we have

$$\text{resistance} = \frac{l}{n}\Sigma\frac{1}{\sigma}, \text{ and } \sigma\lambda = \text{constant} = \kappa, \text{ say,}$$

$$\therefore \Sigma\frac{1}{\sigma} = \frac{1}{\kappa}\Sigma\lambda.$$

But $V = \text{volume} = \frac{l}{n}\Sigma\sigma = \frac{l}{n}\kappa\Sigma\frac{1}{\lambda}$,

$$\therefore \frac{1}{\kappa} = \frac{l}{nV}\Sigma\frac{1}{\lambda},$$

and

$$\Sigma\frac{1}{\sigma} = \frac{l}{nV}\Sigma\lambda\,\Sigma\frac{1}{\lambda}.$$

The correction for the ends of the tube employed by SIEMENS is erroneous, being calculated on the supposition that the divergence of the current takes place from the curved surface of a hemisphere of radius equal to that of the tube. This is tantamount to assuming a constant potential over the solid hemisphere conceived as of infinite conductivity, and gives of course a result too small—R for both ends together. The proper correction, which probably is not of much importance, would depend somewhat upon the mode of connexion of the tube with the terminal cups, but cannot differ much from $\frac{\pi}{2}$R (for both ends), as we have seen. (I have since found that SIEMENS was aware of the small error in this correction.)

section from the line of intersection of the planes. Now $\delta\theta = ds \div \varrho$, if ds be the intercept on the axis between the normal planes, and ϱ the radius of curvature at the point in question. The lower limit to the resistance is thus expressed by

$$\int ds \frac{1}{\iint \frac{\varrho d\sigma}{r}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

In the particular case of a tube of revolution (such as an anchor-ring) $\iint \frac{\varrho d\sigma}{r}$ is a constant, and the limit which now coincides with the true resistance varies as the length of the axis, and is evidently independent of its position. In general the value of the integral will depend on the axis used, but it is in every case less than the true value of the resistance. In choosing the axis, the object is to make the artificial planes of constant potential agree as nearly as possible with the true equipotential surfaces.

A still further generalization is possible by taking for the artificial equipotential surfaces those represented by the equation

$$F = \text{const.}$$

For all systems of surfaces, with one exception, the resistance found on this assumption will be too small. The exception is of course when the surfaces $F=$ const. coincide with the undisturbed equipotential surfaces. The element of resistance between the surfaces F and $F + dF$ is

$$\frac{1}{\iint \frac{1}{dn} d\sigma},$$

where dn is the distance between the surfaces at the element $d\sigma$, and the integration goes over the surface F as far as the edge of the tube. Now

$$dn = dF \div \sqrt{\left(\frac{dF}{dx}\right)^2 + \left(\frac{dF}{dy}\right)^2 + \left(\frac{dF}{dz}\right)^2};$$

$\therefore$ limit to resistance

$$= \int \frac{dF}{\iint d\sigma \sqrt{\left(\frac{dF}{dx}\right)^2 + \left(\frac{dF}{dy}\right)^2 + \left(\frac{dF}{dz}\right)^2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

an expression whose form remains unchanged when $f(F)$ is written for F. If $F = r$, so that the surfaces are spheres,

$$\Sigma \left(\frac{dF}{dx}\right)^2 = \left(\frac{dF}{dr}\right)^2 = 1;$$

$$\therefore \text{ limit } = \int \frac{dr}{\iint d\sigma} = \int \frac{dr}{\sigma}.$$

This form would be suitable for approximately conical tubes, the vertex of the cone being taken as origin of r.

The last formulæ, (55) and (56), are perhaps more elaborate than is required in the present state of acoustical science, and it is rather in the theory of electricity that their interest would lie; but they present themselves so readily as generalizations of previous results that I hope that they are not altogether out of place in the present paper. In all these cases we have the advantage that the quantity sought is determined by a minimum property, and is therefore subject to a much smaller error than exists in the conditions which determine it.

PART III.

Experimental.

The object of this Part is to detail some experiments on resonators instituted with a view of comparing some of the formulæ of Parts I. and II. with observation. HELMHOLTZ in his paper on organ-pipes has compared his own theory with the experiments of SONDHAUSS and WERTHEIM for the case of resonators whose communication with the external atmosphere is by simple holes in their sides. The theoretical result is embodied in (5) and (23), or for circular holes (24) and runs,

$$n=\frac{a}{2\pi}\sqrt{\frac{2R}{S}}=\frac{a}{\pi}\sqrt{\frac{R}{2S}}; \quad \ldots \ldots \ldots \quad (57)$$

or when the area of the opening is approximately circular and of magnitude σ,

$$n=\frac{a\sigma^{\frac{1}{4}}}{2^{\frac{1}{2}}\pi^{\frac{5}{4}}S^{\frac{1}{2}}}. \quad \ldots \ldots \ldots \quad (58)$$

On calculation HELMHOLTZ finds

$$n=56174\,\frac{\sigma^{\frac{1}{4}}}{S^{\frac{1}{2}}}*,$$

the unit of length being metrical.

The empirical formula found by SONDHAUSS is

$$n=52400\,\frac{\sigma^{\frac{1}{4}}}{S^{\frac{1}{2}}},$$

which agrees completely with theory as regards its form, but not so well in the value it assigns to the constant multiplier. The difference corresponds to more than a semitone, and is in the direction that the observed notes are all too low. I can only think of two explanations for the discordance, neither of which seem completely satisfactory. In the first place, SONDHAUSS determined his resonant notes by the pitch of the sound produced when he blew obliquely across the opening through a piece of pipe with a flattened end. It is possible that the proximity of the pipe to the opening was such as to cause an obstruction in the air-passage which might sensibly lower the pitch. Secondly, no account is taken of the thickness of the side of the vessel, the effect of which must be

* The velocity of sound is taken at the freezing-point; otherwise the discordance would be greater.

to make the calculated value of n too great. On the other hand, two sources of error must be mentioned which would act in the opposite direction. The air in the vicinity of the opening must have been sensibly warmer than the external atmosphere, and we saw in Part I. how sensitive resonators of this sort must be to small changes in the physical properties of the gas which occupy the air-passages. Indeed SAVART long ago remarked on the instability of the pitch of short pipes, comparing them with ordinary organ-pipes. The second source of disturbance is of a more recondite character, but not, I think, less real. It is proved in works on hydrodynamics that in the steady motion of fluids, whether compressible or not, an increased velocity is always accompanied by a diminished pressure. In the case of a gas the diminished pressure entails a diminished density. There seems therefore every reason to expect a diminution of density in the stream of air which plays over the orifice of the resonator, which must cause a rise in the resonant note. But independently of these difficulties, the theory of pipes or other resonators made to speak by a stream of air directed against a sharp edge is not sufficiently understood to make this method of investigation satisfactory. For this reason I have entirely abandoned the method of causing the resonators to speak in my experiments, and have relied on other indications to fix the pitch. The only other experiments that I have met with on the subject of the present paper are also by SONDHAUSS, who has been very successful in unravelling the complications of these phenomena without much help from theory*. For flasks with long necks he found the formula

$$n=46705\,\frac{\sigma^{\frac{1}{2}}}{\mathrm{L}^{\frac{1}{2}}\mathrm{S}^{\frac{1}{2}}}$$

as applicable when the necks are cylindrical and not too short, corresponding to the theoretical

$$n=\frac{a}{2\pi}\,\frac{\sigma^{\frac{1}{2}}}{\mathrm{L}^{\frac{1}{2}}\mathrm{S}^{\frac{1}{2}}}, \qquad (59)$$

obtained by combining (5) and (21), or, in numbers with metrical units,

$$n=54470\,\frac{\sigma^{\frac{1}{2}}}{\mathrm{L}^{\frac{1}{2}}\mathrm{S}^{\frac{1}{2}}}.$$

The discrepancy is no doubt to be attributed (at least in great measure) to the omission of the correction to the length of the neck.

In the experiments about to be described the pitch of the resonator was determined in various ways. Some of the larger ones had short tubes fitted to them which could be inserted in the ear. By trial on the piano or organ the note of maximum resonance could be fixed without difficulty, probably to a quarter of a semitone. In most of the experiments a grand piano was used, whose middle c was in almost exact unison with a fork of 256 vibrations per second. Whenever practicable the harmonic undertones were also used as a check on any slight difference which might be possible in the quality of consecutive notes. Indeed the determination was generally easier by means of the

* Pogg. Ann. t. lxxxi.

first undertone (the octave), or even the second (the twelfth), than when the actual note of the resonator was used. The explanation is, I believe, not so much that the overtones belonging to any note on the piano surpass in strength the fundamental tone, although that is quite possible*, as that the ear (or rather the attention) is more sensitive to an increase in the strength of an overtone than of the fundamental. However this may be, there is no doubt that a little practice greatly exalts the power of observation, many persons on the first trial being apparently incapable of noticing the loudest resonance. Another plan very convenient, though not to be used in measurements without caution, is to connect one end of a piece of india-rubber tubing† with the ear, while the other end is passed into the interior of the vessel. In this way the resonance of any wide-mouthed bottle, jar, lamp-globe, &c. may be approximately determined in a few seconds; but it must not be forgotten that the tube in passing through the air-passage acts as an obstruction, and so lowers the pitch. In many cases, however, the effect is insignificant, and can be roughly allowed for without difficulty. For large resonators this method is satisfactory, but in other cases is no longer available. I have, however, found it possible to determine with considerable precision the pitch of small flasks with long necks by simply holding them rather close to the wires of the piano while the chromatic scale is sounded. The resonant note announces itself by a quivering of the body of the flask, easily perceptible by the fingers. Since it is not so easy by this method to divide the interval between consecutive notes, I rejected those flasks whose pitch neither exactly agreed with any note on the piano nor exactly halved the interval. In some cases it is advantageous to sing into the mouth, taking care not to obstruct the passage; the resonant note is recognized partly by the tremor of the flask, and partly by a peculiar sensation in the throat or ear, hard to localize or describe.

The precision obtainable in any of these ways may seem inferior to that reached by several experimenters who have used the method of causing the resonators or pipes to speak by a stream of air. That the apparent precision in the last case is greater I of course fully admit; for any one by means of a monochord could estimate the pitch of a continuous sound within a smaller limit of error than a quarter of a semitone. But the question arises, *what* is it that is estimated? Is it the natural note of the resonator? I have already given my reasons for doubting the affirmative answer; and if the doubt is well grounded, the greater precision is only apparent and of no use theoretically. I may add, too, that many of the flasks that I used could not easily have been made to speak by blowing. If they sounded at all it was more likely to be the first overtone, which is the note rather of the neck than the flask: see equation (20). In carrying out the measurements of the quantities involved in the formula, the volume of the flask or reservoir was estimated by filling it with water halfway up the neck, which was then measured, or in some cases weighed. The measurements of the neck were made in two ways according to the length. Unless very short their capacity was measured by water,

* In this respect pianos, even by the same maker, differ greatly.

† The black French tubing, about $\frac{1}{2}$ inch in external diameter, is the pleasantest to use.

and the expression for the resistance (54) in a simplified form was used. The formula for n then runs,

$$n=\frac{a}{2\pi L}\sqrt{\frac{\text{vol. of neck}}{\text{vol. of flask}\times\left(1+\frac{\pi}{2}\frac{R}{L}\right)}} \quad \ldots\ldots\ldots \quad (60)$$

When, on the other hand, the necks were short, or simply holes of sensible thickness, the following formula was used,

$$n=\frac{a(r_1+r_2)}{4}\sqrt{\frac{1}{\pi LS\left(1+\frac{\pi(r_1+r_2)}{4L}\right)}}, \quad \ldots\ldots\ldots \quad (61)$$

r_1, r_2 being the radii or halves of the diameters as measured at each end. It is scarcely necessary to say that the estimation of pitch was made in ignorance of the theoretical result; otherwise it is almost impossible to avoid a certain bias in dividing the interval between the consecutive notes.

TABLE I.

No. of observation.	S, in cub. centims.	V, in cub. centims.	L, in inches.	R, in inches.	n, by calculation.	n, by observation.	Difference, in mean semitones.
1	805	68	$4\frac{1}{2}$	$\frac{1}{2}$	127·7	126	+·23
2	1350	126·7	$5\frac{5}{8}$	$\frac{5}{8}$	107·7	108·7	−·16
3	7100	430	$\frac{13}{4}$	$\frac{23}{16}$, $\frac{15}{8}$	122·3	120	+·33
4	405	49·9	$3\frac{13}{16}$	$\frac{1}{2}$	179·7	180	−·03
5	180	21·26	$\frac{23}{8}$	$\frac{3}{8}$	233·7	228	+·42
6	785	36·84	$\frac{9}{4}$	$\frac{17}{32}$, $\frac{10}{16}$	174·3	176	−·16
7	210	32·5	$3\frac{7}{8}$	$\frac{13}{32}$	201·9	204	−·18
8	312	29·32	$3\frac{1}{4}$	$\frac{3}{8}$	186·3	182	+·41
10	6300	270	$3\frac{3}{8}$	$\frac{38}{32}$, $\frac{44}{32}$	104·2	102·4	+·29
12	54·69	11·6	$2\frac{5}{16}$	not recorded	391·6	384*	+·34

In Table I. the first column gives the number of the experiment, the second the volume of the reservoirs, including half the necks, the third the volume of the necks themselves, the fourth their lengths, and the fifth their radii measured, when necessary, at both ends. In the sixth column is given the number of vibrations per second calculated from (60), the velocity of sound being taken at 1123 feet per second, corresponding to 60° F., about the temperature of the room in which the pitch was determined.

* 12 was originally estimated an octave too low, so that the number in the Table is the double of what was put down as the result of observation.

Column 7 contains the values of n estimated by means of the pianoforte, while in 8 is given for convenience the discrepancy between the observed and calculated values expressed in parts of a mean semitone.

1, 2, 4, 5, 6, 7, 8, 12 were glass flasks with well-defined nearly cylindrical necks, the body of the flask being approximately spherical. Of these 1 and 2 had small tubes cemented into them, which were inserted in the ear; the pitch of the rest was estimated mainly by their quivering to the resonant note. 3 and 10 were globes intended for burning phosphorus in oxygen gas, and their pitch was fixed principally by the help of the india-rubber tube passed through the neck. A good ear would find no difficulty in identifying the note produced when the body of the globe is struck with the soft part of the hand. The agreement is I think very satisfactory, and is certainly better than I expected, having regard to the difficulties in the measurements of pitch and of the dimensions of the flasks. The average error in Table I. is about a quarter of a semitone, and the maximum error less than half a semitone. It should be remembered that there is no arbitrary constant to be fixed as best suits the observations, but that the calculated value of n is entirely determined by the dimensions of the resonator and the velocity of sound. If a lower value of the latter than 1123 were admissible, the agreement would be considerably improved.

TABLE II.

No. of experiment.	S, in cub. centims.	L, in inches.	d, in inches.	n, by calculation.	n, by observation.	Difference, in mean semitones.
9	1245	$2\frac{11}{16}$	$\frac{14}{16}$, $\frac{15}{16}$	107·3	108	$-$ ·11
11	216·6	$\frac{1}{6}$	1, $\frac{15}{16}$	526	538	$-$ ·39
13	1245	1	1	163·2	170	$-$ ·71
14	1245	$\frac{1}{6}$	1, $\frac{15}{16}$	219·4	213	$+$ ·51
15	3090	$\frac{3}{32}$	$2\frac{1}{16}$	218·1	227·5	$-$ ·73
16	3240	$1\frac{14}{16}$	$1\frac{3}{4}$	131·3	142	$-$1·36
17	3240	$\frac{1}{8}$	$1\frac{1}{8}$, $1\frac{1}{16}$	149·1	153·5	$-$ ·50
18	3240	$\frac{5}{16}$	$1\frac{5}{16}$	153·2	153·5	$-$ ·03
19	3240	$\frac{5}{16}$	1	129·1	132	$-$ ·38
20	3040	neglected	$\frac{32}{48}$	128·6	128	$+$ ·08
21	3240	$1\frac{1}{8}$	$1\frac{1}{16}$, $1\frac{1}{32}$	101·5	103·5	$-$ ·34
22	3240	$\frac{3}{16}$	$2\frac{7}{32}$	216	229	$-$1·01

Table II. contains the results of the comparison between theory and observation for a number of resonators whose necks were too short for the convenient measurement of the volume. The length and diameter were measured with care and used in formula

(61). In 9, 13, 14 the reservoir consisted of the body of a flask whose neck had been cut off close, and which was fitted with a small tube for insertion in the ear. In 9 and 13 there was a short glass or tin tube fitted into the opening*, while in 14 the mouth was covered (air-tight) with a piece of sheet gutta percha pierced by a cork borer; 11 was a small globe treated in the same way. 15 to 22 were all experiments with a globe of a moderator-lamp, which also had a tube for the ear, one opening being closed by a piece of plate glass cemented over it. Sometimes a little water was poured in for greater convenience in determining the pitch, whence the slightly differing values of S. In 15 the opening was clear, and in 16 fitted with a brass tube; in 17 it was covered with a gutta-percha face, in 18, 19, 21 with a wooden face bored by a centre-bit, and in 20 with a piece of tin plate carrying a circular hole; 22 contains the result when the other opening of the globe was used clear.

On inspection of Table II. it appears that the discrepancy between theory and observation is decidedly greater than in Table I., in fact about double, whether we consider the maximum or the mean error. The cause of some of the large errors may, I think, be traced. 13 and 16 had necks of just the length for which the correction $\frac{\pi}{4}$ R may not be quite applicable. A decided flow back round the edge of the outer end must take place with the effect of diminishing the value of α. In order to test this explanation, a piece of millboard was placed over the outer end of the tube in 16 to represent the infinite plane. A new estimation of n, as honest as possible, gave $n=137$, which would considerably diminish the error. I fancied that I could detect a decided difference in the resonance according as the millboard was in position or not; but when the theoretical result is known, the difficulty is great of making an independent observation. In 15 and 22, where the apertures of the globe were used clear, the error is, I believe, due to an insufficient fulfilment of the condition laid down at the commencement of this paper. Thus in 15 the wave-length$=1123\div 227=4\cdot 9$ feet; or $\frac{1}{4}\lambda=1\cdot 2$ feet, which is not large enough compared to the diameter of the globe (6 inches). The addition of a neck lowers the note, and then the theory becomes more certainly applicable.

It may perhaps be thought that the observations on resonance in Tables I. and II. do not extend over a sufficient range of pitch to give a satisfactory verification of a general formula. It is true that they are for the most part confined within the limits of an octave, but it must be remembered that if the theory is true for any resonant air-space, it may be extended to include all *similar* air-spaces in virtue of SAVART's law alone—a law which has its foundations so deep that it hardly requires experimental confirmation. If this be admitted, the range of comparison will be seen to be really very wide, including all *proportions* of L and R. When the pitch is much higher or much lower than in the Tables, the experimental difficulties are increased. For much lower tones the ear is not sufficiently sensitive, while in the case of the higher tones some of the indications relied on to fix the pitch are no longer available.

* Gutta percha softened in hot water is very useful for temporary fittings of this sort.

Some experiments were next made with the moderator globe and two openings. The theoretical formulæ are

$$n=\frac{a}{2\pi}\sqrt{\frac{c_1+c_2}{S}},$$

$$c=\frac{\pi R^2}{L+\frac{\pi}{2}R}.$$

TABLE III.

No. of experiment.	S, in cub. centims.	c_1, in inches.	c_2, in inches.	c_1+c_2, in inches.	n, by calculation.	n, by observation.	Difference, in mean semitones.
23	3240	2·00	1·95	3·95	303·4	320	·92
24	3240	2·00	1·01	3·01	264·8	282	1·09
25	3240	·715	1·01	1·725	200·4	204	·31

In 23 both holes of the globe were clear, and in 24, 25 they were covered with wooden faces carrying holes of various diameters. The error in 23, 24 is to be ascribed to the same cause as in 15 and 22 above.

The last experiment that I shall describe was made in order to test the theory of double resonance, but is not quite satisfactory, for the same reason as 15, 22, 23, 24. Two moderator globes were cemented together so as to form two chambers communicating with each other and with the external air. The natural openings were used clear, and the resonance (which was not very good) was estimated by means of a tube connecting the ear with one of them. The observations gave for the values of n,

$$\left.\begin{array}{l}\text{High note}=384,\\ \text{Low note }=213.\end{array}\right\}$$

The result of calculation from the dimensions of the globes and openings by means of the formulæ of Parts I. and II. was

$$\left.\begin{array}{l}\text{High note}=360,\\ \text{Low note }=212.\end{array}\right\}$$

The error in the high note is about a semitone.

[The two moderator globes were fitted up again as a double resonator, only with bored wooden disks over the holes, so as to lower the note and render the theory more strictly applicable. The pitch was much better defined than before, and gave

$$\left.\begin{array}{l}\text{Low note }=152{\cdot}5,\\ \text{High note}=240.\end{array}\right\}$$

$$c_1=c_3=1{\cdot}008;\quad c_2={\cdot}7152.$$
$$a=1133\ (70^\circ\ \text{F.})$$

Thus

$$n_1=\frac{1133\times 12}{6\cdot 28}\sqrt{\frac{16\cdot 39\times 1\cdot 008}{3200}}=155\cdot 6,$$

$$n_2=\frac{1133\times 12}{6\cdot 28}\sqrt{\frac{16\cdot 39\times 2\cdot 438}{3200}}=241\cdot 9.$$

The agreement is now very good.

One of the outer holes was stopped with a plate of glass. The resonance of the high note was feeble though well defined; that of the low was rather loud but badly defined.

$$\left.\begin{array}{l}\text{The high note was put at } 225\\ \text{,, low ,, ,, } 90\end{array}\right\}.$$

$$S=3150,\quad S'=3250,$$

$$c_2=\cdot 7152,\quad c_1=1\cdot 008,\quad c_1+c_2=1\cdot 7232.$$

Calculating from these data, we get

$$\left.\begin{array}{l}n_1=225\cdot 2,\\ n_2=\ 90\cdot 5.\end{array}\right\}$$

The agreement is here much better than was expected, and must be in part fortuitous.

I will now detail two experiments made to verify the formula marked (20*a*). A moderator chimney was plugged at the lower end with gutta percha, through which passed a small tube for application to the ear. The bulb was here represented by the enlargement where the chimney fits on to the lamp. On measurement,

$$\frac{S}{\sigma}=4\cdot 16 \text{ inches},\quad L=5\cdot 367 \text{ inches},\quad \alpha=\frac{\pi}{4}R=\cdot 471.$$

Thus

$$\tan k\times 9\cdot 611=\frac{1}{k\times 4\cdot 161};$$

from this the value of k was calculated by the trigonometrical tables. Finally,

$$n=251\cdot 4.$$

As the result of observation n had been estimated at 252.

In another case,

$$L=5\cdot 767,\quad \alpha=\cdot 537,\quad \frac{S}{\sigma}=3\cdot 737,\quad n \text{ by observation}=351.$$

The result of calculation is $n=350\cdot 3$. These are the only two instances in which I have tried the formula (20*a*). It is somewhat troublesome in use, but appears to represent the facts very closely; though I do not pretend that the above would be average samples of a large series. There is no necessity for the irregularity at the lower end taking the form of an enlargement. For example, the formula might be applied to a truly cylindrical pipe with a ball of solid material resting at the bottom.]

I had intended to have made these experiments more complete, particularly on multiple resonance, but have not hitherto had time. However, the results obtained seem quite sufficient to establish a substantial agreement between theory and fact. It should be understood that those here presented are not favourable specimens selected out of a large number, but include, with one exception, all the measurements attempted. There are many kinds of bottles and jars, and among them some of the best resonators, which do not satisfy the fundamental condition on which our theory rests. The deductive treatment of the problem in such cases presents great difficulties of a different kind from any encountered in this paper. Until they are surmounted the class of resonators referred to are of no use for an exact comparison between theory and observation, though they may be of great service as aids to investigation in other directions.